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Transient waves generated by a source on a circle

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Abstract. The solutions of the inhomogeneous wave equation are obtained. The source is distributed on a circle at rest or moving with a constant velocity. The wavefunction is found as a series of transient modes by means of incomplete separation of variables, the Riemann formula, and the integrals containing three Bessel functions.

1. Introduction

In this paper we present the explicit solutions of the inhomogeneous wave equation in the spacetime domain. The general time-dependent source is distributed on a circle which is at rest or starts at a fixed moment of time and moves with a constant velocity. Realizations of such sources can vary from their continuous distribution on a circle at rest up to the source pulse moving along a helicoidal line.

The wavefunctions are expressed in terms of transient modes in cylindrical coordinates. The analogous expansions had previously been constructed in spherical coordinates, where the wavefunctions were represented in terms of spherical harmonics. Clapp *et al* (1970) constructed these expansions with the help of the retarded solution of the inhomogeneous equation using previously established addition theorems (Clapp 1970) and integral theorems (Clapp and Li 1970) for the spherical harmonics. Manankova (1972) obtained the explicit spacetime expansion in spherical coordinates by means of the incomplete separation of variables (Smirnov 1937) and the Riemann method. Here we use the above methods and special relation for integrals containing three Bessel functions (Macdonald 1909).

Evidently, the spacetime structure of the source defines an application for possible representations of the wave solutions.

2. Solution of the wave equation

The wave equation in cylindric coordinates ρ, φ, z is

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{1}{\rho^2}\frac{\partial^2}{\partial\varphi^2}+\frac{\partial^2}{\partial z^2}-\frac{\partial^2}{\partial\tau^2}\right]\psi=-\frac{4\pi}{c}j\,. \tag{1}$$

The initial condition is

$$\psi\equiv 0 \qquad \tau<0\,. \tag{2}$$

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Here an arbitrary function j describes the source of the wave disturbance, $j = 0$ for $\tau < 0$, $\tau = ct$, t is time and c is the wavefront velocity (the velocity of light for electromagnetic waves).

We construct the solution of the problem by separating variables φ and ρ. Representing the desired wavefunction and the source in the form

$$\psi(\rho,\varphi,z,\tau) = \sum_{m=-\infty}^{\infty} e^{im\varphi}\psi_m(\rho,\tau,z) \qquad j(\rho,\varphi,z,\tau) = \sum_{m=-\infty}^{\infty} e^{im\varphi} j_m(\rho,\tau,z) \tag{3}$$

we have from (1) and (2) the problem for the expansion coefficients ψ_m

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) + \frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\tau^2} - \frac{m^2}{\rho^2}\right]\psi_m = -\frac{4\pi}{c} j_m \qquad \psi_m = 0 \quad \tau < 0. \tag{4}$$

Making the Fourier–Bessel transform

$$F(s,z,\tau) = \int_0^\infty \mathrm{d}\rho\, \rho J_m(s\rho) F(\rho,z,\tau)$$

$$F(\rho,z,\tau) = \int_0^\infty \mathrm{d}s\, s J_m(s\rho) F(s,z,\tau)$$

where $J_m(s\rho)$ is the Bessel function of the first kind, from (4) one obtains

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial\tau^2} - s^2\right)\psi_m(s,z,\rho) = -\frac{4\pi}{c} j_m(s,z,\tau) \tag{5}$$

$$\psi_m(s,z,\tau) \equiv 0 \qquad \tau < 0. \tag{6}$$

The solution of the above problem can be obtained by the Riemann formula, so

$$\psi_m(s,z,\tau) = \frac{2\pi}{c}\int_0^\tau \mathrm{d}\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z'\, j_m(s,z',\tau') J_0\big(s\sqrt{(\tau-\tau')^2-(z-z')^2}\big).$$

Here $J_0\big(\sqrt{(\tau-\tau')^2-(z-z')^2}\big)$ is the Riemann function for the equation (5) while the expansion coefficients j_m are defined from (3). Making the inverse Fourier–Bessel transform, one can write the above results

$$\psi_m(\rho,z,\tau) = \frac{2\pi}{c}\int_0^\tau \mathrm{d}\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z' \int_0^\infty \mathrm{d}s\, s J_m(s\rho) J_m(sa)$$
$$\times J_0\big(s\sqrt{(\tau-\tau')^2-(z-z')^2}\big) j_m(s,z',\tau'). \tag{7}$$

This expression gives the algorithm to construct the coefficients for the expansion of the wavefunction.

3. Wavefunction expansion for the source at rest

Let the source be distributed on a circle with radius a. Then the function j is

$$j = \frac{\delta(\rho-a)}{\rho}\delta(z) f(z,\varphi,\tau)$$

where $\delta(x)$ is the Dirac distribution. The expansion coefficients of the function j are

$$j_m(\rho,z,\tau) = \frac{\delta(\rho-a)}{\rho}\delta(z) f_m(z,\tau). \tag{8}$$

Substituting (8) into (7) and using the property of the δ-function, one obtains

$$\psi_m(\rho, z, \tau) = \frac{2\pi}{c}\int_0^{\tau - z} \mathrm{d}\tau'\, f(\tau', 0)\int_0^{\infty} \mathrm{d}s\, s J_m(sa) J_m(s\rho) J_0\left(s\sqrt{(\tau-\tau')^2 - z^2}\right). \qquad (9)$$

We denote the integral

$$I_m \equiv \int_0^{\infty} \mathrm{d}s\, s J_m(sa) J_m(s\rho) J_0\big(s\sqrt{(\tau-\tau')^2 - z^2}\big).$$

The dependence of the integral I_m on the variables ρ, τ, z and the parameter a can be expressed as (Gradshteyn and Ryzhik 1969, equation 6.578)

$$I_m = \begin{cases} \dfrac{1}{\rho a} P^{1/2}_{m-1/2}(\cos\vartheta)\dfrac{1}{\sqrt{2\pi\sin\vartheta}} & (\rho - a)^2 < (\tau-\tau')^2 + z^2 < (\rho + a)^2 \\ 0 & \begin{cases} (\tau-\tau')^2 - z^2 < (\rho - a)^2 \\ \text{or} \\ (\tau-\tau')^2 - z^2 > (\rho + a)^2 \end{cases} \end{cases}$$

where $P^{1/2}_{m-1/2}(\cos\vartheta)$ is the Legendre function of the first kind,

$$\cos\vartheta = \frac{1}{2a\rho}\left(\rho^2 + a^2 + z^2 - (\tau-\tau')^2\right)$$

$$\sin\vartheta = \sqrt{1 - \frac{\left(\rho^2 + a^2 + z^2 - (\tau-\tau')^2\right)^2}{4a^2\rho^2}}.$$

Hence the integral I_m is non-zero if

$$\tau - \sqrt{(\rho+a)^2 + z^2} < \tau' < \tau - \sqrt{(\rho-a)^2 + z^2}.$$

Remembering that

$$P^{1/2}_{m-1/2}(\cos\vartheta) = \sqrt{\frac{2}{\pi\sin\vartheta}}\cos m\vartheta$$

we can write I_m as

$$I_m = \frac{1}{\rho a}\frac{\cos m\vartheta}{\pi\sin\vartheta}.$$

From (9) one can obtain

$$\psi_m = \frac{2}{ca\rho}\int_T^{\tau - \sqrt{(\rho-a)^2+z^2}} \mathrm{d}\tau'\, f_m(\tau')\frac{\cos m\vartheta}{\sin\vartheta} \qquad (10)$$

where

$$T = \max\big(0, \tau - \sqrt{(\rho+a)^2 + z^2}\big).$$

Using the Chebyshev polynomials of the first kind $T_m(x) = \cos(m\cos^{-1}x)$ we have for the wavefunction ψ

$$\psi = \frac{2}{ca\rho}\sum_m \mathrm{e}^{im\varphi}\int_T^{\tau-\sqrt{(\rho-a)^2+z^2}} \mathrm{d}\tau'\, f_m(\tau')\frac{T_m\left(\frac{1}{2a\rho}(\rho^2 + a^2 + z^2 - (\tau-\tau')^2)\right)}{\sqrt{1 - \frac{1}{4a^2\rho^2}(\rho^2 + a^2 + z^2 - (\tau-\tau')^2)^2}}. \qquad (11)$$

This relation describes the wavefunction ψ for the arbitrary time dependence of $f_m(\tau')$.

The expansion coefficients ψ_m in the particular case of the δ-pulse source $f_m(\tau) = \delta(\tau)$ are

$$\psi_m = \frac{2}{ca\rho}\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z^2-\tau^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z^2-\tau^2)^2}}\,[h(\tau-\sqrt{(\rho-a)^2+z^2})$$
$$-h(\tau-\sqrt{(\rho+a)^2+z^2})] \tag{12}$$

where

$$h(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases}$$

is the Heaviside function.

From (12) we have for the pulse duration $\delta = \sqrt{(\rho+a)^2+z^2} - \sqrt{(\rho-a)^2+z^2}$. The pulse fronts are surfaces $\tau^2 = (\rho-a)^2+z^2$ and $\tau^2 = (\rho+a)^2+z^2$. The polynomials $T_m(x)$ have m zeros, hence ψ_m has m zeros for $\tau \subset \left[\sqrt{(\rho-a)^2+z^2},\ \sqrt{(\rho+a)^2+z^2}\right]$.

If $f(\tau') = \exp\left(\mathrm{i}\omega\tau'/c\right)$ and $\tau > \sqrt{(\rho+a)^2+z^2}$, one obtains for the steady-state process

$$\psi_m = \frac{2}{ca\rho}\mathrm{e}^{\mathrm{i}\omega\tau/c}\int_{\sqrt{(\rho-a)^2+z^2}}^{\sqrt{(\rho+a)^2+z^2}} \mathrm{d}s\mathrm{e}^{-\mathrm{i}\omega s/c}\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z^2-(\tau-\tau')^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z^2-(\tau-\tau')^2)^2}}\,. \tag{13}$$

In the case $J(\tau') = \cos(\omega/c)\tau'$ one can obtain

$$\psi_m = \frac{2}{ca\rho}[\cos(\omega\tau/c)\Phi_m^{(1)}(\rho,z,a) + \sin(\omega\tau/c)\Phi_m^{(2)}(\rho,z,a)] \tag{14}$$

where

$$\begin{pmatrix}\Phi_m^{(1)}(\rho,z,a)\\ \Phi_m^{(2)}(\rho,z,a)\end{pmatrix} = \int_{\sqrt{(\rho-a)^2+z^2}}^{\sqrt{(\rho+a)^2+z^2}} \mathrm{d}s \begin{pmatrix}\cos(\omega s/c)\\ \sin(\omega s/c)\end{pmatrix}\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z^2-s^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z^2-s^2)^2}}\,.$$

Note that the amplitude coefficients for $m = 0$ are solutions of the Helmholz equation (Bateman 1955).

4. Expansion of the wavefunction for the source distributed on a circle moving with a constant velocity $v < c$

The time-dependent source is distributed on a circle which starts its motion at the time $\tau = 0$ with constant velocity v along the axis z.

$$j = \frac{\delta(\rho-a)}{\rho}\delta(z-\beta\tau)h(z)f(\tau,z,\varphi) \qquad \tau<0 \tag{15}$$

where $\beta = v/c$. Using new variables

$$z_\beta = \frac{z-\beta\tau}{\sqrt{1-\beta^2}} \qquad \tau_\beta = \frac{\tau-\beta z}{\sqrt{1-\beta^2}} \tag{16}$$

we have from (4) the equation for the coefficients $\psi_m(\rho, z_\beta, \tau_\beta)$

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right)+\frac{\partial^2}{\partial z_\beta^2}-\frac{\partial^2}{\partial\tau_\beta^2}-\frac{m^2}{\rho^2}\right]\psi_m$$

$$= -\frac{4\pi}{c}\frac{\delta(\rho-a)}{\rho}h(z_\beta+\beta\tau_\beta)\delta\big(z_\beta\sqrt{1-\beta^2}\big)f_m\left(\frac{\tau_\beta+\beta z_\beta}{\sqrt{1-\beta^2}};\frac{z_\beta+\beta\tau_\beta}{\sqrt{1-\beta^2}}\right). \quad (17)$$

with the initial conditions

$$\psi_m = 0 \qquad \tau_\beta < 0\,. \quad (18)$$

Separating the angular and radial variables and applying the Riemann method we obtain for the function ψ_m

$$\psi_m = \frac{2\pi}{c\sqrt{1-\beta^2}}\int_0^{\tau_\beta}\mathrm{d}\tau'_\beta\int_{\tau'_\beta+z_\beta-\tau_\beta}^{-\tau'_\beta+z_\beta+\tau_\beta}\mathrm{d}z'_\beta\,\delta(z'_\beta)\,f_m\left(\frac{\tau'_\beta+\beta z'_\beta}{\sqrt{1-\beta^2}};\frac{z_\beta+\beta\tau_\beta}{\sqrt{1-\beta^2}}\right)h(z'_\beta+\beta\tau'_\beta)\times$$

$$\times\int_0^\infty \mathrm{d}s\, s\, J_m(s\rho)J_m(sa)J_0\big(s\sqrt{(\tau_\beta-\tau'_\beta)^2-(z_\beta-z'_\beta)^2}\big)\,.$$

Due to the property of the δ-function one obtains

$$\psi_m = \frac{2\pi}{c\sqrt{1-\beta^2}}\int_0^{\tau_\beta-z_\beta}\mathrm{d}\tau'_\beta\, f_m\left(\frac{\tau'_\beta}{\sqrt{1-\beta^2}};\frac{\beta\tau'_\beta}{\sqrt{1-\beta^2}}\right)I_m(\tau'_\beta, z'_\beta)$$

where

$$I_m = \int_0^\infty \mathrm{d}s\, s J_m(s\rho)J_m(sa)J_0\big(s\sqrt{(\tau_\beta-\tau'_\beta)^2-(z_\beta-z'_\beta)^2}\big)\,.$$

The integral I_m is non-zero in the spacetime domain

$$(\rho+a)^2+z_\beta^2 < (\tau_\beta-\tau'_\beta)^2 < (\rho-a)^2+z_\beta^2\,.$$

Using the relations

$$I_m = \frac{1}{\rho a}\frac{\sqrt{2}}{\pi}\frac{\cos(m\vartheta_\beta)}{\sin\vartheta_\beta}$$

$$\cos\vartheta_\beta = \frac{1}{2a\rho}(\rho^2+a^2+z_\beta^2-(\tau_\beta-\tau'_\beta)^2)$$

$$\sin\vartheta_\beta = \sqrt{1-\frac{1}{4a^2\rho^2}\big(\rho^2+a^2+z^2-(\tau_\beta-\tau'_\beta)^2\big)^2}$$

we have the solution of the problem (17) and (18)

$$\psi_m = \frac{2}{ca\rho}\frac{1}{\sqrt{1-\beta^2}}\int_T^{\tau_\beta-\sqrt{(\rho-a)^2+z_\beta^2}}\mathrm{d}\tau'_\beta\, f_m\left(\frac{\tau'_\beta}{\sqrt{1-\beta^2}},\frac{\beta\tau'_\beta}{\sqrt{1-\beta^2}}\right)\frac{\cos m\vartheta_\beta}{\sin\vartheta_\beta} \quad (19)$$

where

$$T = \max\big(0, \tau_\beta - \sqrt{(\rho+a)^2+z_\beta^2}\big)\,.$$

According to the definition of the polynomials $T_m(x)$ we can write the following for the wavefunction ψ:

$$\psi = \frac{2}{ca\rho}\frac{1}{\sqrt{1-\beta^2}}\sum_{m=-\infty}^{\infty} e^{im\varphi}\int_T^{\tau_\beta-\sqrt{(\rho-a)^2+z_\beta^2}} d\tau'_\beta\, f_m\left(\frac{\tau'_\beta}{\sqrt{1-\beta^2}}, \frac{\beta\tau'_\beta}{\sqrt{1-\beta^2}}\right)$$
$$\times\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z_\beta^2-(\tau_\beta-\tau'_\beta)^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z_\beta^2-(\tau_\beta-\tau'_\beta)^2)}}. \tag{20}$$

In the case where the δ-pulse $f_m(\tau') = \delta(\tau')$ and $f_m(\tau'_\beta) = \sqrt{1-\beta^2}\,\delta(\tau'_\beta)$, we have from (19)

$$\psi_m = \frac{2}{ca\rho}\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z_\beta^2-\tau_\beta^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z_\beta^2-\tau_\beta^2)^2}}\left[h\left(\tau_\beta-\sqrt{(\rho-a)^2+z_\beta^2}\right)\right.$$
$$\left.-h\left(\tau_\beta-\sqrt{(\rho+a)^2+z_\beta^2}\right)\right] \tag{21}$$

where $\tau^2 - z^2 = \tau_\beta^2 - z_\beta^2$, hence the expressions (12) for the source being at rest and (21) for the moving source are equivalent.

Let $f_m(\tau', 0) = e^{i\omega\tau'/c}$ and $\tau_\beta > \sqrt{(\rho+a)^2+z_\beta^2}$. From (21) one can obtain for the expansion coefficients ψ_m (the steady-state process)

$$\psi_m = \frac{2}{ca\rho}\frac{1}{\sqrt{1-\beta^2}} e^{i\frac{\omega}{c}\frac{\tau_\beta}{\sqrt{1-\beta^2}}}\int_{\sqrt{(\rho-a)^2+z_\beta^2}}^{\sqrt{(\rho+a)^2+z_\beta^2}} ds\,\exp\left(-i\frac{\omega}{c}\frac{s}{\sqrt{1-\beta^2}}\right)$$
$$\times\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z_\beta^2-s^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z_\beta^2-s^2)^2}}. \tag{22}$$

Hence, for $f_m(\tau') = \cos(\omega\tau'/c)$ we have

$$\psi_m = \frac{2}{ca\rho}\frac{1}{\sqrt{1-\beta^2}}\left[\cos\left(\frac{\omega}{c}\frac{1}{\sqrt{1-\beta^2}}\tau_\beta\right)\Phi_m^{(1)} + \sin\left(\frac{\omega}{c}\frac{1}{\sqrt{1-\beta^2}}\tau_\beta\right)\Phi_m^{(2)}\right] \tag{23}$$

where

$$\begin{pmatrix}\Phi_m^{(1)}\\ \Phi_m^{(2)}\end{pmatrix} = \int_{\sqrt{(\rho-a)^2+z_\beta^2}}^{\sqrt{(\rho+a)^2+z_\beta^2}} ds \begin{pmatrix}\cos\left(\frac{\omega}{c}\frac{1}{\sqrt{1-\beta^2}}s\right)\\ \sin\left(\frac{\omega}{c}\frac{1}{\sqrt{1-\beta^2}}s\right)\end{pmatrix}\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z_\beta^2-s^2)\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z_\beta^2-s^2)}}.$$

Being expressed via the variables τ, z , this solution represents a steady-state process, which differs from the result for the source at rest (14).

5. Expansion of the wavefunction for the source distributed on a circle moving with the velocity of light

This is a special case of the problem. The source starts at the moment of time $\tau = 0$ and moves with the velocity of light c along the axis z. In this case the expansion coefficients for the function j are

$$j_m = \frac{\delta(\rho-a)}{\rho}\delta(z-\tau)h(z)f_m(\tau, z) \tag{24}$$

and after separation of the radial variable we have the following problem from (4):

$$\left(\frac{\partial^2}{\partial z^2} - \frac{\partial^2}{\partial \tau^2} - s^2\right)\psi_m(s,z,\tau) = -\frac{4\pi}{c} f_m(\tau,z) J_m(sa) h(z)\delta(z-\tau) \tag{25}$$

$$\psi_m(s,z,\tau) \equiv 0 \qquad \tau < 0. \tag{26}$$

The solution of this problem is

$$\psi_m(\rho,z,\tau) = \frac{2\pi}{c}\int_0^\tau \mathrm{d}\tau' \int_{\tau'+z-\tau}^{-\tau'+z+\tau} \mathrm{d}z' \, f_m(\tau',z') \int_0^\infty \mathrm{d}s\, s J_m(sa) J_m(s\rho)$$
$$\times J_0\left(s\sqrt{(\tau-\tau')^2-(z-z')^2}\right)\delta(z'-\tau')h(z'). \tag{27}$$

Hence one obtains

$$\psi_m = \frac{2\pi}{c}\int_0^{(\tau+z)/2} \mathrm{d}\tau' \, f_m(\tau',\tau') \int_0^\infty \mathrm{d}s\, s J_m(sa) J_m(s\rho) J_0\big(s\sqrt{(\tau-\tau')^2-(z-\tau')^2}\big).$$

The internal integral I_m is non-zero when τ' is in the range

$$\frac{\tau^2 - z^2 - (\rho+a)^2}{2(\tau-z)} < \tau' < \frac{\tau^2 - z^2 - (\rho-a)^2}{2(\tau-z)}. \tag{28}$$

Then

$$I_m = \frac{1}{\rho a}\frac{\cos m\vartheta}{\pi \sin\vartheta}$$

where

$$\cos\vartheta = \frac{1}{2a\rho}\left(\rho^2 + a^2 + z^2 - \tau^2 + 2(\tau-z)\tau'\right)$$

$$\sin\vartheta = \sqrt{1 - \frac{1}{4a^2\rho^2}\left(\rho^2 + a^2 + z^2 - \tau^2 + 2(\tau-z)\tau'\right)^2}.$$

We have for the expansion coefficients of the wavefunction

$$\psi_m = \frac{2}{ca\rho}\int_T^{[\tau^2-z^2-(\rho-a)^2]/2(\tau-z)} \mathrm{d}\tau' f_m(\tau',\tau')\frac{\cos m\vartheta}{\sin\vartheta} \tag{29}$$

where

$$T = \max\left(0, \frac{\tau^2 - z^2 - (\rho+a)^2}{2(\tau-z)}\right).$$

Writing the expression for ψ_m with the help of the Chebyshev polynomials one can obtain

$$\psi = \frac{2}{ca\rho}\sum_m \mathrm{e}^{\mathrm{i}m\varphi}\int_T^{[\tau^2-z^2-(\rho-a)^2]/2(\tau-z)} \mathrm{d}\tau' f_m(\tau',\tau')$$
$$\times\frac{T_m\left(\frac{1}{2a\rho}(\rho^2+a^2+z^2-\tau^2+2(\tau-z)\tau')\right)}{\sqrt{1-\frac{1}{4a^2\rho^2}(\rho^2+a^2+z^2-\tau^2+2(\tau-z)\tau')^2}} \tag{30}$$

for the expansion of the wavefunction (3). The above expression gives the represantation ψ for the arbitrary time dependence of the source.

The result (30) in the particular case $m = 1$ is

$$\psi_1 = \frac{2}{c\rho a}\int_T^{[\tau^2-z^2-(\rho-a)^2]/2(\tau-z)} \mathrm{d}\tau'\, f_1(\tau',\tau')\frac{\frac{1}{2a\rho}(\rho^2+a^2+z^2-\tau^2+2(\tau-z)\tau')}{\sqrt{1-\frac{1}{4a^2\rho^2}\left(\rho^2+a^2+z^2-\tau^2+2(\tau-z)\tau'\right)^2}}\,.$$

When the function f_m = constant, one obtains

$$\psi_1 = \begin{cases} \frac{\pi}{c\rho a(\tau-z)}\sqrt{\tau^2-z^2-(\rho+a)^2}\sqrt{\tau^2-z^2-(\rho-a)^2} & z^2+(\rho-a)^2 < \tau^2 < z^2+(\rho+a)^2 \\ 0 & \tau^2 > z^2+(\rho+a)^2\,. \end{cases}$$

In this case for $\tau^2 > z^2 + (\rho + a)^2$ we have the following expansion

$$\psi = \frac{2}{c(\tau-z)}\sum_m \mathrm{e}^{\mathrm{i}m\varphi}\int_{-1}^{1}\mathrm{d}x\,\frac{T_m(x)}{\sqrt{1-x^2}}\,. \tag{31}$$

Here

$$x = \frac{1}{2a\rho}\left(\rho^2+a^2+z^2-\tau^2+(\tau-z)\tau'\right)\,. \tag{32}$$

Using the orthogonal property of the Chebyshev polynomials one obtains $\psi_m = 0$ for $m \neq 0$ when $\tau^2 > z^2 + (\rho + a)^2$ and $\psi_m \neq 0$ if $\tau^2 < z^2 + (\rho + a)^2$. Hence the pulse duration is $\delta(\rho, z, \tau) = 2a\rho/(\tau - z)$ for every expansion coefficient ψ_m , $m \neq 0$. The expressions $(\rho-a)^2+z^2+2(\tau-z)\tau' = 0$, $(\rho+a)^2+z^2+2(\tau-z)\tau' = 0$ describe the front surfaces of ψ_m in the case $m \neq 0$ and $f_m(\tau', \tau')$ = constant.

We have the important case when a cosinusoidal wave is turned on in the circle: $f_m(\tau', \tau') = \cos\left(\omega\tau'/c\right)$. Making the substitution (32) and using equation 7.355 of Gradshteyn and Ryzhik (1969), we get for the expansion coefficient ψ_m (the case of the steady-state mode)

$$\begin{aligned} \psi_{2n} &= (-1)^n\frac{2\pi}{c(\tau-z)}\cos\left(\frac{\omega}{2c}\left(\tau+z-\frac{\rho^2+a^2}{\tau-z}\right)\right)J_{2n}\left(\frac{\omega}{c}\frac{a\rho}{\tau-z}\right) && m = 2n \\ \psi_{2n+1} &= (-1)^n\frac{2\pi}{c(\tau-z)}\sin\left(\frac{\omega}{2c}\left(\tau+z-\frac{\rho^2+a^2}{\tau-z}\right)\right)J_{2n+1}\left(\frac{\omega}{c}\frac{a\rho}{\tau-z}\right) && m = 2n+1 \end{aligned} \tag{33}$$

where n is an integer. Note that the expressions (12) can be obtained from (30) for the δ-pulse source $f_m = \delta(\tau)$.

6. Discussion

The alternative wavefunction representation is determinated by the structure of the source. Let us obtain the expansion of the wavefunction in terms of spherical harmonics and modes in cylindrical coordinates for the axisymmetric circumferential source being at rest. The corresponding wave equation in spherical coordinates r, ϑ, φ is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial\psi}{\partial\vartheta}\right)-\frac{\partial^2\psi}{\partial\tau^2} = -\frac{4\pi}{c}j$$

where

$$j = \frac{1}{r^2}\delta(r-r_0)\delta(\vartheta-\vartheta_0)f(\tau) \qquad j = 0 \quad \tau < 0\,.$$

Here r_0, ϑ_0 are the source coordinates and $a = r_0 \sin \vartheta_0$ is the radius of the circumferential source. The initial condition is

$$\psi = 0 \qquad \tau < 0 .$$

Using the expansions

$$\psi = \sum_{n=0}^{\infty} \widetilde{\psi_n}(\tau, r) P_n(\cos\vartheta) \qquad j = \sum_{n=0}^{\infty} \widetilde{j_n}(\tau, r) P_n(\cos\vartheta)$$

where $P_n(\cos\vartheta)$ is the Legendre polynomial, one can obtain the problem for the expansion coefficients $\widetilde{\psi_m}$

$$\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial \tau^2} - \frac{n(n+1)}{r^2}\right)\widetilde{\psi_n} = -\frac{4\pi}{c}\widetilde{j_n} \qquad \widetilde{\psi_n} = 0 \quad \tau < 0 .$$

The expansion coefficients for the source are

$$\widetilde{j_n} = (n + \tfrac{1}{2}) P_n(\cos\vartheta_0) \frac{1}{r^2} \delta(r - r_0) f(\tau) .$$

The solution of the above problem can be found with the help of the Riemann formula (Manankova 1972). For the wavefunction ψ we have

$$\psi(\tau, z, \vartheta) = \frac{2\pi}{c r r_0} \sum_{n=0}^{\infty} (n + \tfrac{1}{2}) P_n(\cos\vartheta) P_n(\cos\vartheta_0) \times \int_{T_1}^{T_2} \mathrm{d}\tau' f(\tau')\, P_n\left(\frac{r^2 + r_0^2 - (\tau - \tau')^2}{2 r r_0}\right) . \tag{34}$$

Here

$$T_1 = \max(0-,\ \tau - r - r_0) \qquad T_2 = \begin{cases} \tau - r + r_0 & \text{if } r > r_0 \\ \tau + r - r_0 & \text{if } r < r_0 . \end{cases}$$

A simpler result can be obtained for the representation of the wavefunction as a series of modes in cylindrical coordinates. Using (11), one obtains

$$\psi = \psi_0 = \frac{2}{c a \rho} \int_T^{\tau - \sqrt{(\rho - a)^2 + (z - z_0)^2}} \mathrm{d}\tau' f(\tau') \frac{1}{\sqrt{1 - \frac{1}{4a^2\rho^2}(\rho^2 + a^2 + (z - z_0)^2 - (\tau - \tau')^2)^2}}$$

where

$$T = \max\left(0-,\ \tau - \sqrt{(\rho + a)^2 + (z - z_0)^2}\right)$$

and $z_0 = a \cot\vartheta_0$. The wavefunction ψ is a pulse confined by the wavefront $\tau^2 = (\rho - a)^2 + (z - z_0)^2$. Note that every expansion coefficient of spherical harmonics (33) is a pulse confined by the spherical fronts $\tau - r + r_0 = 0,\ r > r_0$ or $\tau + r - r_0 = 0,\ r < r_0$ and only the sum of the terms gives us the correct result.

7. Conclusion

The above solutions of the scalar wave equation can be applied to electromagnetic waves. For example, we obtain from Maxwell's equations

$$\nabla^2 E_i - \frac{\partial^2}{\partial \tau^2} E_i = \frac{4\pi}{c}\left(\frac{\partial}{\partial \tau} j + c \operatorname{grad} \rho\right)_i$$

$$\nabla^2 B_i - \frac{\partial^2}{\partial \tau^2} B_i = \frac{4\pi}{c} (\operatorname{rot} j)_i \,.$$

Here E_i and B_i are the Cartesian components of the electric field strength and the magnetic induction vectors, and ρ and j are the electric charge and current densities. Gaussian units are used. We have the wave equation (1), where ψ is E_i or B_i and j_i is Cartesian component of vectors $\left(\frac{\partial j}{\partial \tau} + c \operatorname{grad} \rho\right)$ or $\operatorname{rot} j$.

Electromagnetic waves can also be described in terms of one-component vector $\mathbf{\Pi} = e_z \Pi$ for the special case $j = e_z j_z$. Maxwell's equations can be satisfied if

$$E_\rho = \frac{\partial^2 \Pi}{\partial \rho \partial z} \qquad E_\varphi = \frac{1}{\rho}\frac{\partial^2 \Pi}{\partial z \partial \varphi} \qquad E_z = \frac{\partial^2 \Pi}{\partial z^2} - \frac{\partial^2 \Pi}{\partial \tau^2}$$

$$B_\rho = \frac{1}{\rho}\frac{\partial^2 \Pi}{\partial \varphi \partial \tau} \qquad B_\varphi = -\frac{\partial^2 \Pi}{\partial \rho \partial \tau} \qquad B_z = 0 \,.$$

Here Π is the Whittaker–Bromwich scalar potential (Nisbet 1955). For $\partial \Pi / \partial \tau$ the inhomogeneous equation (1), where j is the z-component of current density j_z, is obtained.

Thus, the present results permit us, in principle, to describe the electromagnetic wave produced by different source pulses, up to the electron moving along a helicoidal line from the fixed moment of time, in terms of the transient modes in cylindric coordinates. Note that case $v = c$ can be used as a simplified model for the description of the electromagnetic fields produced by a pulse of hard radiation (Karzas and Latter 1965, Longmire 1978). In particular, the above electromagnetics model is important in connection with the relatively new problem of formation of the localized electromagnetic pulses which are akin to Brittingham's focus wave modes.

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